# Surface Integrals and Flux

**Problem 1** (Stewart, Exercises 16.5.(13,15,18)). *Determine whether or not the following vector fields are conservative. For each conservative, find a function $f$ such that $F = \nabla f$.*

*(1)* $F(x, y, z) = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$.

*(2)* $F(x, y, z) = \langle z \cos y, 2xyz^3, xz \sin y, x \cos y \rangle$.

*(3)* $F(x, y, z) = \langle e^x \sin yz, ze^x \cos yz, ye^x \cos yz \rangle$.

**Problem 2** (Stewart, Exercises 16.5.(19,20)). *Is there a vector field $G$ on $\mathbb{R}^3$ satisfying the following condition.*

*(1)* $\mathrm{curl}(G) = \langle x \sin y, \cos y, z - xy \rangle$. *Explain why?*

*(2)* $\mathrm{curl}(G) = \langle x, y, z \rangle$. *Explain why?*

The **Laplace operator** $\nabla^2$ is defined by $\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$, where $f$ is a function having continuous second partial derivatives. The Laplace operator can be applied also to a vector field $F = \langle P, Q, R \rangle$ as follows: $\nabla^2 F = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle$.

**Problem 3** (Stewart, Exercises 16.5.(27,28,29)). *For vector fields $F$ and $G$ on $\mathbb{R}^3$, argue the following identities:*

*(1)* $\mathrm{div}(F \times G) = G \cdot \mathrm{curl}\, F - F \cdot \mathrm{curl}\, G$,

*(2)* $\mathrm{div}(\nabla F \times \nabla G) = 0$,

*(3)* $\mathrm{curl}(\mathrm{curl}\, F) = \mathrm{grad}(\mathrm{div}\, F) - \nabla^2 F$.

**Problem 4** (Stewart, Exercises 16.5.(30,31)). *For the vector field $r = \langle x, y, z \rangle$, argue the following identities:*

*(1)* $\nabla \cdot (|r| r) = 4|r|$,

*(2)* $\nabla^2(|r|^3) = 12r$,

*(3)* $\nabla(1/|r|) = -r/|r|^3$,

*(4)* $\nabla(\ln |r|) = r/|r^2|$.

**Problem 5** (Stewart, Exercises 16.5.(33,34)). *Let $C$ be a positively-oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$ given by $r(t) = \langle x(t), y(t) \rangle$ with $a \leq t \leq b$, and let $n(t) = 1/|r'(t)| \langle y'(t), -x'(t) \rangle$.*

*(1) Use Green's Theorem to argue that*

$$\oint_C F \cdot n\, ds = \iint_{\mathrm{int}(C)} \mathrm{div}\, F(x, y)\, dA,$$

*for any smooth vector field $F$ on $\mathbb{R}^2$.*

*(2) Use the previous part to argue Green's First Identity:*

$$\iint_{\text{int}(C)} f\nabla^2 g\, dA = \oint_C f(\nabla g)\cdot n\, ds - \iint_{\text{int}(C)} \nabla f\cdot\nabla g\, dA,$$

*for any functions $f$ and $g$ whose appropriate partial derivatives exist and are continuous.*

*(3) Use the previous part to argue Green's Second Identity:*

$$\iint_{\text{int}(C)} (f\nabla^2 g - g\nabla^2 f)\, dA = \oint_C (f\nabla g - g\nabla f)\cdot n\, ds,$$

*for any functions $f$ and $g$ whose appropriate partial derivatives exist and are continuous.*

**Problem 6** (Stewart, Exercise 16.6.34)**.** *Find the equation of the tangent plane to the surface parameterized by $r(u,v) = (u^2+1, v^3+1, u+v)$ at $(5,2,3)$.*

**Problem 7** (Cal Final, Summer 2018W)**.** *Find the tangent plane to the parametrized surface $r(u,v) = \langle u^2-1, uv, v^3\rangle$ at the point $(3,4,8)$.*

**Problem 8** (Stewart, Exercise 16.6.61)**.** *Find the area of the part of the sphere $x^2+y^2+z^2 = 4z$ that lies inside the paraboloid $z = x^2+y^2$.*

**Problem 9.** *Deduce the formula for the flux of a vector field $F = \langle P, Q, R\rangle$ across a surface $S$ that is the graph of a smooth function $g\colon U\subset\mathbb{R}^2\to\mathbb{R}$.*

**Problem 10** (Stewart, Exercises 16.7.(15,26))**.** *Evaluate the following surface integrals.*

*(1) $\iint_S y\, dS$, where $S$ is the surface $y = x^2+4z$ for $(x,z)\in[0,1]\times[0,1]$.*

*(2) $\iint_S F\cdot dS$, where $F(x,y,z) = \langle y, -x, 2z\rangle$ and $S$ is the hemisphere $x^2+y^2+z^2 = 4$ and $z\geq 0$ oriented downward.*

**Problem 11** (Stewart, Exercise 16.7.43)**.** *A fluid has density 870 $kg/m^3$ and flows with velocity $v(x,y,z) = \langle z, y^2, x^2\rangle$, where $x, y,$ and $z$ are measured in meters and the components of $v$ in meters per second. Find the rate of flow outward through the cylinder $x^2+y^2 = 4$ for $0\leq z\leq 1$. [Hint: the rate of flow outward is the flux $\iint_S(\rho v)\cdot n\, dS$.]*

**Problem 12** (Stewart, Exercise 16.7.49)**.** *Let $F$ be an inverse square field, that is, $F(r) = cr/|r|^3$ for some constant $c$, where $r = \langle x,y,z\rangle$. Show that the flux of $F$ across a sphere $S$ with center the origin is independent of the radius of $S$.*

**Problem 13** (Cal Final, Fall 09)**.** *Evaluate the surface integral*

$$\iint_S z\, dS,$$

*where $S$ is the part of the sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder $x^2+y^2 = 1$ and above the $xy$-plane. [Answer: $2\pi$]*

**Problem 14** (UC Berkeley Final)**.** *Evaluate the flux*

$$\iint_T F \cdot dS$$

*of the vector field* $F(x, y, z) = \langle -x, xy, zx \rangle$ *across the triangle* $T$ *with vertices* $(1, 0, 0)$, $(0, 2, 0)$, *and* $(0, 0, 2)$ *with downward orientation. [Answer:* $1/3$*]*

**Problem 15** (UC Berkeley Final)**.** *Let* $S$ *be a surface which is contained in the plane* $z = x+y$, *oriented upward. Suppose that* $S$ *has area* $2018$. *Consider the constant vector field* $F = \langle 3, 4, 5 \rangle$. *Calculate the flux of* $F$ *across the surface* $S$. *[Answer:* $-4036/\sqrt{3}$*]*

## References


[1] J. Stewart: *Calculus* 8th Edition, Cengage Learning, Boston 2016.